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On the Numerical Methods for the Singular Parabolic Equations in Fluid Dynamics

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The concern here is with the numerical methods for the well-known singular parabolic equations in unsteady boundary-layer flows behind a moving shock and the development of boundary-layers on a semi-infinite plate impulsively set into motion. These equations can be derived with a semisimilar transformation by which the domains of solution are mapped on to [0, 1]. It is shown that this class of singular parabolic equations can be elegantly and accurately solved by using the successive-overrelaxation method normally applied to elliptic equations. No artificial viscosity or numerical attenuation is required. Details of the computational procedures are given. Salient natures of the numerical method are analyzed. Results of the analysis show that as far as the stability of computations is concerned, the sign of the diffusivity is not important; however, to obtain convergent solution, the "initial" conditions have to be specified consistently.

1. INTRODUCTION

Well-posed parabolic equations, encountered mostly in the problems involving diffusion process, have positive diffusivity. Their solutions are uniquely determined by the given boundary conditions and *an* initial condition [1]. Numerically, this class of equations is generally solved by marching in the direction of the timelike (parabolic) variable. Parabolic equations with negative diffusivity (in the mathematical sense) are classified as ill-posed problems. Solutions for this class of parabolic equations are discussed in [2].

There is another class of "parabolic" equations, which result from the transformation of variables, having characteristics different from those equations mentioned above. Classical examples are equations from semisimilar formulations of the problems in unsteady boundary layer on a semi-infinite plate impulsively set into motion (Stewartson's problem) [3]; the Stewartson's problem with generalized starting condition by Cheng [4]; and the unsteady compressible boundary layer behind a moving shock on a flat plate by Lam and Crocco [5].

Two features distinguish this class of "parabolic" equations from the classical well-posed parabolic equations and the ill-posed parabolic equations: (1) In part of the domain of solution, the signs of the diffusivity (in the mathematical sense) are mixed and (2) two "initial" conditions are specified by the two ends of the timelike

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independent variable. Parabolic equations of this kind are called singular parabolic equations [5]. (These equations are different from the elliptic–parabolic equations studied by Franklin and Rodemich [6] in the sense that in the latter the "initial" conditions are specified by different portion of the two "initial" stations.) According to Ref. [5], two "initial" conditions are admissible because of the existence of the domain where the signs of the diffusivity are mixed. For the convenience of discussion, we shall refer to this part of the domain of solution the mixed region.

Obtaining solutions in the mixed region has been shown to be difficult. Numerical methods for solving the singular parabolic equations that arise from the Stewartson's problem have been presented only recently by Dennis [7] and Williams and Rhyne [8]. Both methods started from one initial station where the diffusivity is positive using the classical marching technique. In the mixed region they applied one-sided backward or forward differencing, depending, respectively, on the diffusivity being positive or negative, for the derivatives with respect to the timelike independent variable. When the forward differencing was used, all the dependent variables at the solution station and stations ahead were not known until the other "initial" station was reached, where another "initial" condition was specified. These unknowns were initially guessed. Iterations in both timelike and spatial directions were performed until the convergence criterion was met. Dennis [7] applied one-sided two-point differencing. Artificial damping was required to make the iteration converge. The effect of this added artificial damping term was shown to make the parabolic equations elliptic. Williams and Rhyne [8] applied one-sided three-point differencing. Because of the nonlinear nature of the problem, the signs of the diffusivity in the mixed region were not a priori known, and had to be determined at each grid point in every iteration so that the direction of the differencing could be adjusted accordingly. Nevertheless, results obtained by both methods showed good agreement with the computation of Hall [9] where the semisimilar transformation was not applied. A good account of this problem has now been given by Telionis [10].

Piquet [11] presented a numerical method applicable to the compressible Stewartson's problem as well as the unsteady compressible boundary-layer flows behind a moving shock. Following the approach of Lam and Crocco [5], he first wrote the boundary-layer equations in terms of Crocco's variable and then applied a semisimilar transformation, which yielded a set of singular parabolic equations. After the solutions were obtained at both ends of the mixed region, the solution inside the mixed region was obtained using a time-dependent scheme by adding to the equations a term of first derivative with respect to an artificial time. The derivatives with respect to the original timelike independent variable were replaced by central differencing. Artificial "attenuation" (which can be shown to correspond to a numerical viscosity) was required to stabilize the computation.

Numerical solution for the singular parabolic equation from the unsteady boundary-layer flow in the shock tube had also been attempted earlier by Ban and Kuerti [12] and Walker and Dennis [13].

It has been shown [14] that the unsteady compressible boundary-layer flow behind a moving shock and the compressible Stewartson's problem can be formulated, without using Crocco's transformation, with a semisimilar transformation by which the domains of solution are mapped on to [0, 1]. This mapping is particularly useful for the numerical computation. Applicabilities of the transformation are not limited to ideal gas or particular temperature-dependent transport coefficients. The resulting equations are singular parabolic in nature. The two "initial" conditions are by themselves the solutions of the governing equations of motion at the respective "initial" stations.

In this paper it will be shown that this class of singular parabolic equations can be elegantly and accurately solved by the successive overrelaxation method normally applied to the numerical solutions of elliptical partial differential equations [15]. Second-order central differencings are applied to both the timelike derivatives and the spatial derivatives. The same differencing scheme and relaxation procedure are applied in the entire domain of solution. There is no need to divide the domain of solution into regions. No artificial damping or attenuation is required.

By an exploratory analysis, some salient natures of the relaxation method applied to this class of singular parabolic equations are revealed. The convergence of the solution of the finite difference equation to that of the differential equation is shown. The relaxation method presented for solving the singular parabolic equations of this type is shown to correspond to a time-dependent scheme for a boundary-value problem. As far as the stability is concerned, the sign of the diffusivity in the original equation does not play an important role. However, to obtain the convergent solution, it is crucial that the two initial conditions are specified consistently.

The numerical procedures are given in great detail in Section 3. Only the solution of the incompressible Stewartson's problem will be presented here for the purpose of comparison. The exploratory analysis on the salient natures of the numerical procedure is given in Section 4. For the sake of completeness, a singular parabolic equation arising from the Stewartson's problem is recapitulated in Section 2.

2. A Singular Parabolic Equation in Unsteady Boundary-Layer Flows

A semi-infinite plate immersed in a quiescent fluid of infinite extent is impulsively set into motion at time $t = 0$ with constant velocity U_e. In a Cartesian coordinate system, with the origin fixed at the leading edge of the plate (Fig. 1), the equations governing the unsteady incompressible boundary-layer flow on the flat plate are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

where x and y are the axes parallel and normal, respectively, to the plate; u and v are

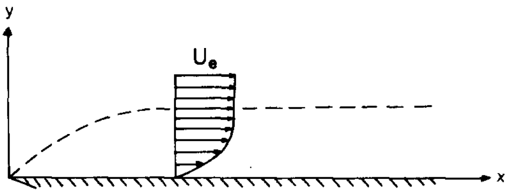

FIG. 1. Coordinate systems for the unsteady boundary-layer flow on a semi-infinite plate impulsively set into motion.

velocity components of fluid in the x and y directions, respectively; and ν is the kinematic viscosity.

Mathematically, the boundary conditions and the initial condition are

for $x \geq 0$ and $t \geq 0$,

$$u = v = 0 \quad \text{at} \quad y = 0, \qquad u \to U_e \quad \text{as} \quad y \to \infty; \tag{3}$$

for $t \geq 0$ and $y > 0$,

$$u = U_e \quad \text{at} \quad x = 0; \tag{4}$$

for $x \geq 0$ and $y > 0$,

$$u = U_e \quad \text{at} \quad t = 0. \tag{5}$$

However, based on physical reasoning [3], the "boundary" condition (4) is replaced by the condition that the flow field at $x/U_e t = 0$ be given by the steady-state Blasius solution. The initial condition (5) is replaced by the condition that as $x/U_e t \to \infty$, the flow field be represented by the Rayleigh solution for the viscous flow on an impulsively started infinite plate. The semisimilar variable $x/U_e t$ is originally given by Stewartson [3].

An alternative semisimilar transformation for this problem has been given by Williams and Rhyne [8] and extended to compressible flow by Wang [14]. The independent variables (x, y, t) are transformed to (ξ, η) by

$$\xi = 1 - e^{-U_e t/x}, \qquad \eta = y/[\nu x(1 - e^{-U_e t/x})/U_e]^{1/2}, \tag{6}$$

and the nondimensional stream function $f(\xi, \eta)$ is given by

$$\psi(x, y, t) = [U_e \nu x(1 - e^{-U_e t/x})]^{1/2} f(\xi, \eta), \tag{7}$$

where $\psi(x, y, t)$ is the stream function related to u and v, by

$$u(x, y, t) = \frac{\partial \psi}{\partial y}, \qquad v(x, y, t) = -\frac{\partial \psi}{\partial x}. \tag{8}$$

In terms of the stream function ψ, the continuity equations (2) is automatically satisfied and the momentum equation (1) can be written as

$$A_0 \frac{\partial \bar{u}}{\partial \xi} = \frac{\partial^2 \bar{u}}{\partial \eta^2} + A_1 \frac{\partial \bar{u}}{\partial \eta}, \tag{9}$$

where

$$A_0 = \xi(1-\xi)[1 + \ln(1-\xi)\bar{u}], \tag{10}$$

$$A_1 = \tfrac{1}{2}[\xi + (1-\xi)\ln(1-\xi)]f + \xi(1-\xi)\ln(1-\xi)\frac{\partial f}{\partial \xi} + (1-\xi)\frac{\eta}{2}, \tag{11}$$

and

$$\bar{u} = \frac{\partial f}{\partial \eta} = \frac{u}{U_e}. \tag{12}$$

The boundary conditions (3) are now given by

$$f = \bar{u} = 0 \quad \text{at} \quad \eta = 0; \qquad \bar{u} \to 1 \quad \text{as} \quad \eta \to \infty. \tag{13}$$

As discussed earlier, at $x = 0$, $\bar{u}$ is given by the steady-state Blasius solution; i.e., the solution of

$$\frac{d^2\bar{u}}{d\eta^2} + \frac{1}{2} f \frac{d\bar{u}}{d\eta} = 0 \tag{14}$$

subject to the boundary conditions given by Eq. (13). At $x = 0$, $\xi = 1$. Equation (14) is obtained from Eq. (9) by setting $\xi = 1$.

At $t = 0$, $\bar{u}$ is given by the Rayleigh solution; i.e., the solution of

$$\frac{d^2\bar{u}}{d\eta^2} + \frac{1}{2} \eta \frac{d\bar{u}}{d\eta} = 0 \tag{15}$$

subject to the boundary conditions given by Eq. (13). At $t = 0$, $\xi = 0$. Equations (15) is obtained from Eq. (9) by setting $\xi = 0$.

Equation (9) is to be solved in the domain $0 \le \eta < \infty$, $0 \le \xi \le 1$, and is formally parabolic with ξ serving as the timelike independent variable. There are two "initial" conditions to be satisfied: one at $\xi = 0$ and the other at $\xi = 1$. In $1 \ge \xi > 1 - e^{-1}$, the sign of the diffusivity A_0 can be either positive or negative and cannot be predeterminated. Parabolic equations of this type are called singular parabolic equations [5].

3. Numerical Method and Results

The nonlinear singular parabolic equation (9) is to be solved subject to the boundary conditions given by Eq. (13) and the two "initial" conditions given, respectively, by the solution of Eqs. (14) and (15).

Using the second-order central difference for both derivatives with respect to ξ and η, Eq. (9) can be written as

$$\bar{u}_{i,j} = [(\bar{u}_{i,j+1} + \bar{u}_{i,j-1})/\Delta\eta^2 + A_1(i,j)(\bar{u}_{i,j+1} - \bar{u}_{i,j-1})/(2\,\Delta\eta) - A_{01}(i)(\bar{u}_{i+1,j} - \bar{u}_{i-1,j})/(2\,\Delta\xi)]/B(i,j), \tag{16}$$

where i and j are the grid index in the ξ and η directions, respectively; $\xi_i = (i-1)\,\Delta\xi$; $\eta_j = (j-1)\,\Delta\eta$; and

$$B(i,j) = 2/\Delta\eta^2 + A_{02}(i)(\bar{u}_{i+1,j} - \bar{u}_{i-1,j})/(2\,\Delta\xi), \tag{17}$$

$$A_{01}(i) = \xi_i(1-\xi_i), \tag{18}$$

$$A_{02}(i) = \xi_i(1-\xi_i)\ln(1-\xi_i), \tag{19}$$

$$A_1(i,j) = \tfrac{1}{2}[\xi_i + (1-\xi_i)\ln(1-\xi_i)]\,f(i,j) + (1-\xi_i)\,\eta_j/2 + \xi_i(1-\xi_i)\ln(1-\xi_i)[f(i+1,j) - f(i-1,j)]/(2\,\Delta\xi), \tag{20}$$

$$f(i,j) = \int_0^{\eta_j} \bar{u}(\xi_i,\eta)\,d\eta. \tag{21}$$

Although solution for Eq. (15) can be obtained analytically in terms of the complimentary error function, it is obtained numerically in this study using the second-order central differencing. The solution is used as the initial guess for the numerical solution of Eq. (14), which is also solved by the second-order central differencing with convergence criterion $\varepsilon = 10^{-6}$. The convergence criterion is defined by Eq. (25).

With the "initial" conditions known, the difference equation (16) is solved by the successive-overrelaxation method [15]. It should be noticed that this method is considered pertinent to the numerical solution of the elliptic equations.

The initial guess is obtained by linear interpolation from the two "initial" conditions.

The iteration proceeds in the direction of increasing j. For each j the procedure proceeds in the direction of increasing i. In the iteration the old values of the dependent variables are replaced immediately by the new values computed except $f(i,j)$, which is recomputed after a relaxation cycle for $\bar{u}$ is completed.

Let n denote the cycle of relaxation; the procedure for solving Eq. (16) is performed by

$$\bar{u}^{n+1}_{i,j} = (1-p)\,\bar{u}^n_{i,j} + p[(\bar{u}^n_{i,j+1} + \bar{u}^{n+1}_{i,j-1})/\Delta\eta^2 + A_1(i,j) \times (\bar{u}^n_{i,j+1} - \bar{u}^{n+1}_{i,j-1})/(2\,\Delta\eta) - A_{01}(i)(\bar{u}^n_{i+1,j} - \bar{u}^{n+1}_{i-1,j})/(2\,\Delta\xi)]/B^{n+1}(i,j), \tag{22}$$

where

$$B^{n+1}(i, j) = 2/\Delta\eta^2 + A_{02}(i)(\bar{u}^n_{i+1,j} - \bar{u}^{n+1}_{i-1,j})/(2\,\Delta\xi) \tag{23}$$

and p, the relaxation factor, is a constant. In $A_1(i, j)$, the $f(i, j)$ is computed by

$$f(i, j) = \int_0^{\eta_j} \bar{u}^n(\xi_i, \eta)\, d\eta \tag{24}$$

using the trapezoid rule.

The convergence criterion is applied to the ratio defined by

$$\varepsilon_{ij} = |(\bar{u}^{n+1}_{i,j} - \bar{u}^n_{i,j})/\bar{u}^n_{i,j}|. \tag{25}$$

The computation is considered to be converged when $\sup(\varepsilon_{ij})$ is less than a preset small value ε. In this study, the test points are set at $i = 2, 4, 6,\ldots, j = 2, 7, 12\ldots$.

Earlier, for the purpose of numerical computations of the inviscid flows in turbomachinery with mixed subsonic and supersonic flow regions, an alternative formulation had been made [16] based on the work of Wu [17]. The governing equations were made formally elliptic by adding to both sides of the equations terms of second derivatives. Inspired by this formulation, we also make a numerical experiment by adding to both sides of Eq. (9) a term $\alpha\, \partial^2\bar{u}/\partial\xi^2$, where α is a constant. The resulting difference equation is

$$\begin{aligned} \bar{u}^{n+1}_{i,j} = (1-p)\,\bar{u}^n_{i,j} + p[(\bar{u}^n_{i,j+1} + \bar{u}^{n+1}_{i,j-1})/\Delta\eta^2 \\ + A^n_1(i, j)(\bar{u}^n_{i,j+1} - \bar{u}^{n+1}_{i,j-1})/(2\,\Delta\eta) + 2\alpha\bar{u}^n_{i,j}/\Delta\xi^2 \\ - A_{01}(i)(\bar{u}^n_{i+1,j} - \bar{u}^{n+1}_{i-1,j})/(2\,\Delta\xi)]/\bar{B}^{n+1}(i, j), \end{aligned} \tag{26}$$

where

$$\bar{B}^{n+1}(i, j) = B^{n+1}(i, j) + 2\alpha/\Delta\xi^2. \tag{27}$$

Equation (26) is reduced to Eq. (22) if α is set to zero.

TABLE I

Rate of Convergence

	$\alpha = 0$			$\alpha = 1$		
ε	$p = 1.4$	$p = 1.6$	$p = 1.8$[a]	$p = 4.0$	$p = 4.4$	$p = 4.6$[b]
10^{-4}	$n = 294$	227	157	243	191	159
10^{-5}	$n = 605$	383	194	424	265	197

[a] Iteration fails to converge for $p = 2.0$.
[b] Iteration fails to converge for $p = 4.8$.

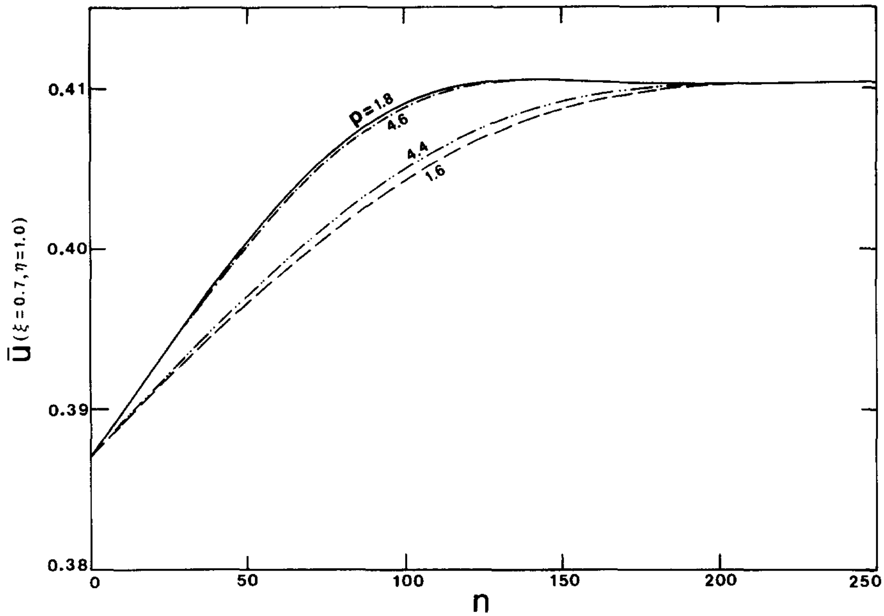

FIG. 2. Converging sequence: — and ---, $\alpha = 0$; -·- and -··-, $\alpha = 1.0$.

The results presented in this paper are based on the computation of $\Delta\xi = 0.05$, $\Delta\eta = 0.0625$, 21 grid points in the ξ direction and 192 grid points in the η direction.

Cycles of iteration required for the solution to converge to different levels of ε are shown in Table I. The typical sequence of convergence, represented by the solution at $i = 15$, $j = 17$ ($\xi = 0.7$, $\eta = 1.0$) is shown in Fig. 2. As expected the rate of convergence depends on p; however, the final solution does not.

From this result it is seen that for the present problem the procedure of adding the α term, with $\alpha = 1$, to both sides of the equation to make the equation formally elliptical does not yield computational advantage. As expected, both procedures, $\alpha = 0$ and $\alpha = 1$, converge to the same result.

Typical velocity profiles $\bar{u}(\xi, \eta)$ are shown in Fig. 3. The profiles for $\xi = 0$ and 1 correspond, respectively, to those of the Raleigh solution and the Blasius solution.

Let C_f be the skin friction coefficient defined by the shear stress on the plate divided by ρU_e^2, then

$$R_x^{1/2} C_f = \frac{1}{\sqrt{\xi}} \left(\frac{\partial \bar{u}}{\partial \eta}\right)_{\eta=0}, \tag{28}$$

where $R_x = U_e x/\nu$. Comparisons of the values of $R_x^{1/2} C_f$ obtained from the present calculation with those calculated by Dennis [7] and Hall [9] are shown in Fig. 4 and in Table II. The tabulated results presented by Hall [9] and Dennis [7] are at the discrete points of $\tau = U_e t/x$. These points do not coincide with the present

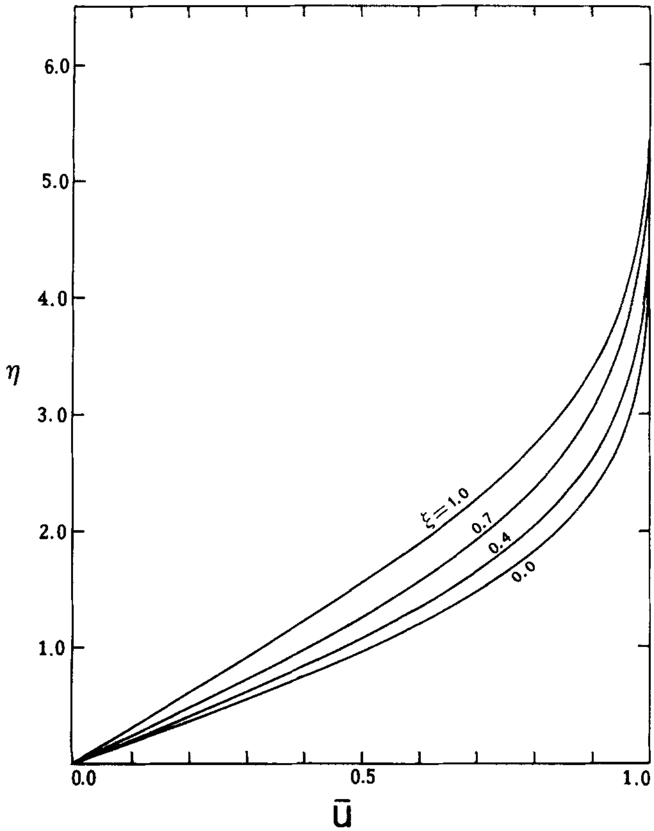

FIG. 3. Velocity profiles for an incompressible boundary-layer on a semi-infinite plate impulsively set into motion.

computational grid points. Third-degree (four points) Lagrangian interpolation is applied to the present result to obtain the solution at these points. These solutions are identified in the Table. The present results agree with those of [7] and [9] to at least the third effective digit.

Dennis [7] assumed $\tau = 1$ as an "initial" station with the velocity profile given by the Rayleigh solution. He integrated the equation of motion from $\tau = 1$ toward the leading edge ($\tau \gg 1$) with the method described earlier in Section 1 and found that the departure of his calculated skin friction from that given by the Rayleigh solution was hardly noticeable until $\tau \geqslant 1.4$. It is shown in Table II that the present calculation agrees with his finding.

In the domain $0 < \tau < 1$ the present result also agrees with the Rayleigh solution at least to the third digit. The difference on the fourth digit could be incurred by the truncation error. Among the sources of truncation errors is the numerical calculation of Eq. (28). The leading term of the truncation error is $\frac{1}{3}[\partial^3 \bar{u}/\partial \eta^3]_{\eta=0} \Delta\eta^2/\sqrt{\xi}$. With

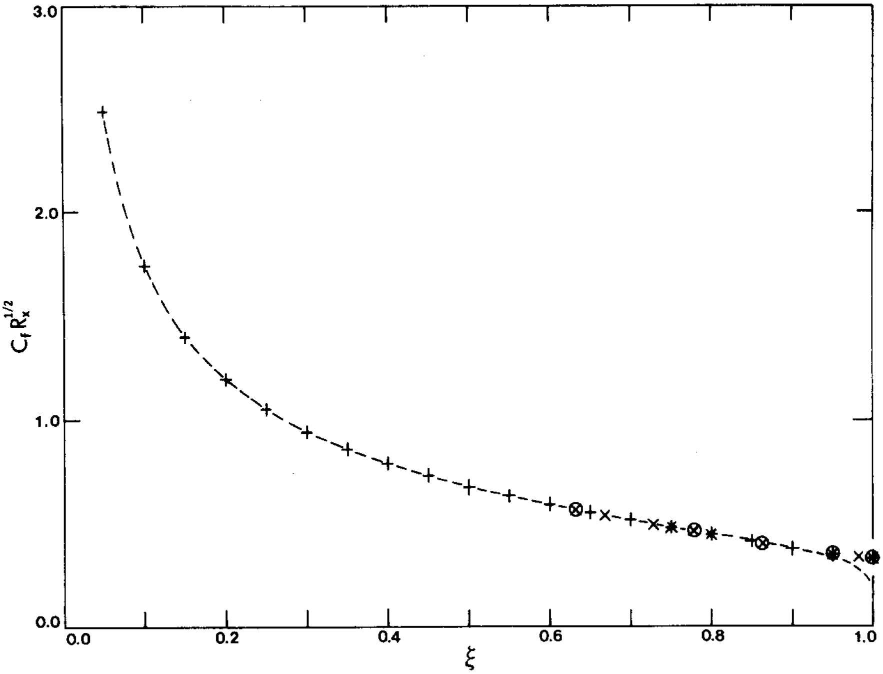

FIG. 4. Skin friction coefficient C_f from Rayleigh solution, ---; present result, +; Ref. [7], ×; Ref. [9], O.

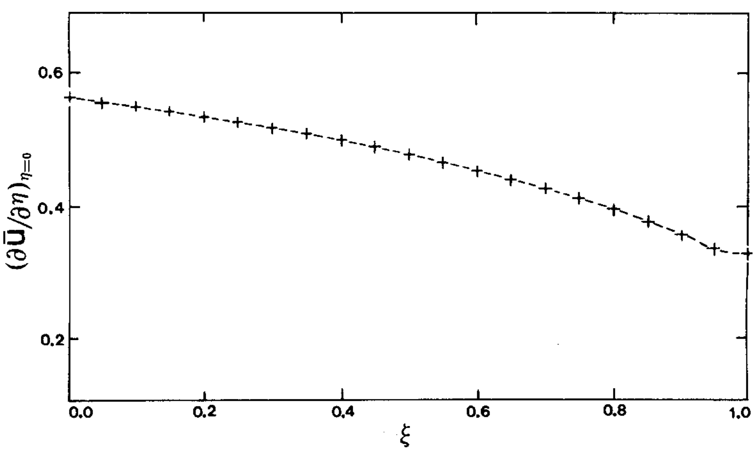

FIG. 5. Comparison of $(\partial \bar{u}/\partial \eta)_{\eta=0}$: present result, +; Ref. [8], ---.

TABLE II

Comparison of Results

ξ	$\tau = U_e t/x$	$R_x^{1/2}C_f$			
		Present	Rayleigh	Dennis [7]	Hall [9]
0.05	0.05129	2.4932	2.4912		
0.10	0.10536	1.7396	1.7382		
0.15	0.16252	1.4006	1.3995		
0.20	0.22314	1.1953	1.1944		
0.25	0.28768	1.0527	1.0519		
0.30	0.35667	0.94541	0.94470		
0.35	0.43078	0.86024	0.85960		
0.40	0.51083	0.79000	0.78938		
0.45	0.59784	0.73023	0.72968		
0.50	0.69315	0.67815	0.67766		
0.55	0.79851	0.63183	0.63137		
0.60	0.91629	0.58987	0.58940		
0.63212	1.0	0.56467[a]	0.56419	0.56420	0.5642
0.65	1.04982	0.55112	0.55064		
0.66713	1.10	0.53840[a]	0.53793	0.53794	
0.69881	1.20	0.51546[a]	0.51503	0.51504	
0.70	1.20398	0.51461	0.51418		
0.72747	1.30	0.49514[a]	0.49483	0.49485	
0.75	1.38629	0.47947	0.47918		
0.75340	1.40	0.47714[a]	0.47683	0.47691	
0.77687	1.50	0.46116[a]	0.46066	0.46090	0.4610
0.79810	1.60	0.44688[a]	0.44603	0.44657	
0.80	1.60944	0.44561	0.44472		
0.85	1.89712	0.41231	0.40962		
0.86466	2.0	0.40277[a]	0.39894	0.4026	0.4025
0.90	2.30259	0.37989	0.37181		
0.95	2.99573	0.34702	0.32597		
0.95021	3.0	0.34691[a]	0.32574	0.3495	0.3493
0.98168	4.0	0.33431[a]	0.28210	0.3347	0.3345
0.99752	6.0	0.33211[a]	0.23033	0.3321	0.3320
0.99966	8.0	0.33207[a]	0.19947	0.3321	0.3320
1.0	∞	0.33207			

[a] Interpolated value.

$\tau = 0.05129$, $\Delta\eta = 0.0625$, and $\bar{u}$ given by the Rayleigh solution, this error is -0.0065. The difference of the present calculation from the Rayleigh solution is 0.0020. In this case, the difference in fourth digit is expected.

Agreement of the present calculated $(\partial\bar{u}/\partial\eta)_{\eta=0}$ with that of [8] is shown in Fig. 5. The two solutions are indistinguishable.

It should be noticed that in the domain $0 < \tau < 1$, A_0 of Eq. (9) is positive. Instead of using a one-sided backward differencing as presented in [7] and [8] in accordance with the concept of numerical method for the *well-posed parabolic equations*, the present method applies the second-order central difference for $\partial\bar{u}/\partial\xi$.

4. Analysis of the Numerical Method

Consider the linear partial differential equation

$$c_1 \frac{\partial \phi}{\partial x} + c_2 \phi = \frac{\partial^2 \phi}{\partial y^2} + c_3 \frac{\partial \phi}{\partial y}, \tag{29}$$

where c_1, c_2, and c_3 can be functions of x and y. Using the numerical method presented in the previous section, with $\alpha = 0$, the corresponding difference equation is

$$\phi_{i,j}^{n+1} = (1-p)\,\phi_{i,j}^{n} + p\left[\phi_{i,j+1}^{n} + \phi_{i,j-1}^{n+1} + \frac{c_3\,\Delta y}{2}\,(\phi_{i,j+1}^{n} - \phi_{i,j-1}^{n+1}) - \frac{c_1\,\Delta y^2}{2\,\Delta x}\,(\phi_{i+1,j}^{n} - \phi_{i-1,j}^{n+1})\right]\Big/(2 + c_2\,\Delta y^2), \tag{30}$$

where c_1, c_2, and c_3 are evaluated at (x_i, y_i).

Equation (30) corresponds to the finite difference equation for solving the initial boundary-value problem

$$\bar{a}\frac{\partial \phi}{\partial t^*} = \frac{\partial^2 \phi}{\partial y^2} + c_3 \frac{\partial \phi}{\partial y} - c_2\phi - c_1\frac{\partial \phi}{\partial x}, \tag{31}$$

where $\bar{a} = (2 + c_2\,\Delta y^2)\,\Delta t^*/(p\,\Delta y^2)$, with the differencing scheme

$$\frac{\partial \phi}{\partial t^*} = \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^{n}}{\Delta t^*}, \tag{32}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\phi_{i,j+1}^{n} - 2\phi_{i,j}^{n} + \phi_{i,j-1}^{n+1}}{\Delta y^2}, \tag{33}$$

$$c_2\phi = c_2\phi_{i,j}^{n}, \tag{34}$$

$$c_3\frac{\partial \phi}{\partial y} = c_3\frac{\phi_{i,j+1}^{n} - \phi_{i,j-1}^{n+1}}{2\,\Delta y}, \tag{35}$$

$$c_1\frac{\partial \phi}{\partial x} = c_1\frac{\phi_{i+1,j}^{n} - \phi_{i-1,j}^{n+1}}{2\,\Delta x}. \tag{36}$$

The solution of Eq. (29) is given by the steady-state solution of Eq. (31).

Notice that Eq. (31) is first order in x. If two boundary conditions for x are specified, it is expected that the solution will exist only if they are consistent. In the case of Eq. (9), the specified conditions for $\bar{u}$ at $\xi = 0$ and 1 are consistent in the sense that they are by themselves the solution of Eq. (9) at the respective location.

To consider the numerical stability (in the von Neumann sense), assume that the solution at $t^* = n\,\Delta t^*$ is given by

$$\phi^n_{i,j} = g^n e^{\hat{i}(i\beta\Delta x + j\gamma\Delta y)}, \tag{37}$$

where $g^n = g\,(n\,\Delta t^*)$ is complex; β and γ are wave numbers; and $\hat{i} = \sqrt{-1}$.

Substitution of Eq. (37) into Eq. (30) gives

$$\frac{g^{n+1}}{g^n} = 1 - \frac{a + b\hat{i}}{c + d\hat{i}}, \tag{38}$$

where

$$a = 2|1 - \cos(\gamma\,\Delta y)| + c_2\,\Delta y^2, \tag{39}$$

$$b = -c_3\,\Delta y\sin(\gamma\,\Delta y) + \frac{c_1\,\Delta y^2}{\Delta x}\sin(\beta\,\Delta x), \tag{40}$$

$$c = \frac{2 + c_2\,\Delta y^2}{p} - \left(1 - \frac{c_3\,\Delta y}{2}\right)\cos(\gamma\,\Delta y) - \frac{c_1\,\Delta y^2}{2\,\Delta x}\cos(\beta\,\Delta x), \tag{41}$$

$$d = \left(1 - \frac{c_3\,\Delta y}{2}\right)\sin(\gamma\,\Delta y) + \frac{c_1\,\Delta y^2}{2\,\Delta x}\sin(\beta\,\Delta x). \tag{42}$$

The amplification factor is given by

$$\left|\frac{g^{n+1}}{g^n}\right|^2 = \frac{(c-a)^2 + (d-b)^2}{c^2 + d^2}. \tag{43}$$

For small p, Eq. (43) can be written as

$$\left|\frac{g^{n+1}}{g^n}\right|^2 = \frac{(2 + c_2\,\Delta y^2)^2 - 2p(2 + c_2\,\Delta y^2)(a + c^*) + O(p^2)}{(2 + c_2\,\Delta y^2)^2 - 2p(2 + c_2\,\Delta y^2)\,c^* + O(p^2)}, \tag{44}$$

where

$$c^* = \left(1 - \frac{c_3\,\Delta y}{2}\right)\cos(\gamma\,\Delta y) + \frac{c_1\,\Delta y^2}{2\,\Delta x}\cos(\beta\,\Delta x). \tag{45}$$

If c_2 is positive, so is a. Since p can be arbitrarily small, it is seen from Eq. (44) that a sufficient (but not necessary) condition for $|g^{n+1}/g^n| \leqslant 1$ is $c_2 \geqslant 0$. Implied in

TABLE III

Comparison of Numerical Solution with Exact Solution

y	$x = 0$ Boundary condition	$x = 0.1$ Exact	$x = 0.1$ Numerical	$x = 0.3$ Exact	$x = 0.3$ Numerical	$x = 0.5$ Exact	$x = 0.5$ Numerical	$x = 0.7$ Exact	$x = 0.7$ Numerical	$x = 0.9$ Exact	$x = 0.9$ Numerical	$x = 1.0$ Boundary condition
0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.75	0.08538	0.09436	0.09488	0.11525	0.11604	0.14076	0.14164	0.17193	0.17274	0.20999	0.21042	0.23208
1.50	0.07773	0.08590	0.08607	0.10492	0.10519	0.12815	0.12843	0.15653	0.15675	0.19118	0.19128	0.21129
2.25	0.05767	0.06374	0.06378	0.07785	0.07792	0.09509	0.09515	0.11614	0.11618	0.14185	0.14187	0.15677
3.00	0.04059	0.04486	0.04487	0.05479	0.05480	0.06692	0.06693	0.08174	0.08174	0.09983	0.09983	0.11033
3.75	0.02811	0.03106	0.03107	0.03794	0.03795	0.04634	0.04634	0.05660	0.05660	0.06913	0.06913	0.07640
4.50	0.01936	0.02140	0.02140	0.02614	0.02614	0.03193	0.03193	0.03900	0.03899	0.04763	0.04763	0.05264
5.25	0.01332	0.01472	0.01472	0.01798	0.01798	0.02196	0.02196	0.02682	0.02682	0.03276	0.03276	0.03621
6.00	0.00916	0.01012	0.01012	0.01236	0.01236	0.01510	0.01510	0.01844	0.01844	0.02252	0.02252	0.02489
6.75	0.00629	0.00696	0.00696	0.00850	0.00850	0.01038	0.01038	0.01267	0.01268	0.01548	0.01548	0.01711
7.50	0.00433	0.00478	0.00478	0.00584	0.00584	0.00713	0.00713	0.00871	0.00871	0.01064	0.01064	0.01176
8.25	0.00297	0.00329	0.00329	0.00401	0.00401	0.00490	0.00490	0.00599	0.00599	0.00731	0.00731	0.00808
9.00	0.00204	0.00226	0.00226	0.00276	0.00276	0.00337	0.00337	0.00411	0.00412	0.00503	0.00503	0.00555
9.75	0.00140	0.00155	0.00155	0.00190	0.00190	0.00232	0.00232	0.00283	0.00283	0.00345	0.00345	0.00382
10.50	0.00097	0.00107	0.00107	0.00130	0.00130	0.00159	0.00159	0.00194	0.00194	0.00237	0.00237	0.00262
11.25	0.00066	0.00073	0.00073	0.00090	0.00090	0.00109	0.00109	0.00134	0.00134	0.00163	0.00163	0.00180
12.00	0.00046	0.00050	0.00050	0.00062	0.00062	0.00075	0.00075	0.00092	0.00092	0.00112	0.00112	0.00124
12.75	0.00031	0.00035	0.00035	0.00042	0.00042	0.00052	0.00052	0.00063	0.00063	0.00077	0.00077	0.00085
13.50	0.00022	0.00024	0.00024	0.00029	0.00029	0.00036	0.00036	0.00043	0.00043	0.00053	0.00053	0.00059
14.25	0.00015	0.00016	0.00016	0.00020	0.00020	0.00024	0.00024	0.00030	0.00030	0.00036	0.00036	0.00040
15.00	0.00010	0.00011	0.00011	0.00014	0.00014	0.00017	0.00017	0.00020	0.00021	0.00025	0.00025	0.00028
15.75	0.00007	0.00008	0.00008	0.00009	0.00009	0.00012	0.00012	0.00014	0.00014	0.00017	0.00017	0.00019
16.50	0.00005	0.00005	0.00005	0.00006	0.00007	0.00008	0.00008	0.00010	0.00010	0.00012	0.00012	0.00013
17.25	0.00003	0.00004	0.00004	0.00004	0.00004	0.00005	0.00005	0.00007	0.00007	0.00008	0.00008	0.00009
18.00	0.00002	0.00003	0.00003	0.00003	0.00003	0.00004	0.00004	0.00005	0.00005	0.00006	0.00006	0.00006
18.75	0.00002	0.00002	0.00002	0.00002	0.00002	0.00003	0.00003	0.00003	0.00003	0.00004	0.00004	0.00004
19.50	0.00001	0.00001	0.00001	0.00002	0.00002	0.00002	0.00002	0.00002	0.00002	0.00003	0.00003	0.00003
20.25	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00002	0.00002	0.00002
21.00	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
21.75	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
22.50	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00001	0.00001	0.00001
23.25	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
24.00	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

this result is that as far as the stability is concerned the sign of c_1 is not important if p is chosen small.

Equation (29), with $c_1 = -\sqrt{x}$, $c_2 = -(1-\sqrt{x})$, and $c_3 = 2.5$, is solved in the domain $x = [0, 1]$ and $y = [0, \infty)$. The boundary conditions are $\phi(x, 0) = \phi(x, \infty) = 0$ and the "initial" conditions are

$$\phi(0, y) = A_1 \exp(-c_3 y/2) \sinh(\gamma_1 y) \tag{46}$$

and

$$\phi(1, y) = A_2 \exp(-c_3 y/2) \sinh(\gamma_2 y), \tag{47}$$

where $\gamma_1 = \gamma_2 = (c_3^2/4 - 1)^{1/2}$, $A_2 = 1.0$, and

$$A_1 = A_2 \exp(-1). \tag{48}$$

The agreement of the numerical result with the exact solution is shown in Table III. In this calculation, $\Delta x = 0.05$, $\Delta y = 0.15$, $p = 0.8$, and the convergence criterion $\varepsilon = 10^{-5}$.

The leading term of the truncation error for the finite difference approximation of $\partial^2\phi/\partial y^2$ is $-\frac{1}{12}\,\partial^4\phi/\partial y^4\, \Delta y^2$. At $x = 0.5$, $y = 0.75$, using the exact solution for ϕ, this truncation error is $+0.0015$. As shown in Table III, the difference between the numerical solution and the exact solution at this point is $+0.0009$.

The two initial conditions specified in this calculation are consistent in the following sense. Both Eqs. (46) and (47) satisfy the differential equation (31). Because of the existence of Eq. (48), they are not independent equations. If one is considered as the initial condition, the other is *the* solution. Attempt to obtain a convergent solution when A_1 is not related to A_2 by Eq. (48), as expected, was not successful.

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